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UNDOMINATED DIRECTIONS IN SIMPLE DYNAMIC GAMES

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## 1. INTRODUCTION AND SUMMARY

Equilibriums to simple games, such as majority rule, in multidimensional spaces require such severe symmetry of preferences as to rarely exist.<sup>1</sup> Consequently, social processes may usually be in disequilibrium. The way they shift the state of the world through time can only be understood when an explicit dynamic mechanism or institution allows sequences of social decisions to be examined.

To date, sequential simple games have been investigated in the context of two "disequilibrium" hypotheses regarding the interconnection between outcomes. Cohen [1977], McKelvey [1976], [1977] and Schofield [1977a] essentially assume that an outcome in one period can be any alternative preferred to the previous outcome by a winning coalition. They show that sequences of outcomes required to satisfy only this dominance property do not satisfy any regularity condition, since such a sequence connects almost any two alternatives in the social choice space. Kramer [1975], in the majority rule context, strengthens their assumption to requiring that an outcome receive a maximal number of votes against the previous outcome. He finds

that these "maximally dominating" sequences always enter the minimax set [Simpson, 1969] when each voter has Euclidean preferences, that is, utility that decreases with the Euclidean distance from an ideal point. This convergence is a regularity property that may provide insight into political situations where mobile challengers oppose fixed incumbents.

In this paper a different hypothesis relating sequential social outcomes is advanced, motivated by the supposition that social change is not instantaneous. More specifically, in any time period only alternatives a small distance from the previous outcome are assumed to be feasible. Taken to its logical and mathematically tractable extreme, this assumption converts the problem into one involving a continuum of social decisions, each of which determines a direction in which to marginally shift the current status quo.<sup>2</sup>

Since social decisions in this setting are directions, the application of cooperative game theory requires that directional preferences be determined from the location of the status quo and the underlying preferences over social states. Directions are represented as vectors of zero or unit length, and in section 2 one direction is said to be preferred to another if it is nearer one's utility gradient evaluated at the status quo. The set of winning coalitions is then used to define a dominance relation on directions, and undominated directions are predicted outcomes to the game. In other words, the status quo is predicted to shift in a direction to which no winning coalition unanimously prefers another direction of shift.

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The distinction between the undominated direction hypothesis and the hypothesis that each chosen point dominates the preceeding one should be emphasized. The continuous version of the latter requires the shift direction to dominate only the zero direction that corresponds to a null shift. The undominated hypothesis, on the other hand, requires the shift direction to be undominated in the set of all directions. Neither hypothesis implies the other, as can be seen in the examples of section 2.

In Appendix A, which supplements section 2, an alternative directional core is defined via an inducement of directional preferences that is independent of utility gradients. This core is found to be contained in the one defined in section 2. The two are identical if each utility function satisfies a condition we label local symmetry.

For comparative purposes, the local point core so often studied since Plott [1967] is examined in section 3. As it too is defined via utility gradients, a second definition involving small neighborhoods is explored in Appendix B. The relationship between the two local point cores is found to be strictly analogous to that between the two directional cores uncovered in Appendix A. Furthermore, the first point core and sometimes the second can be defined in terms of the analogous directional cores. Specifically, it is shown that a point is locally undominated if and only if the zero direction corresponding to a null shift is undominated. This result is strengthened in section 3 when its point core is shown to consist of points whose directional cores contain all directions. Finally,

section 3 concludes with the demonstration that in the benchmark case of Euclidean preferences, directions that "point" to an existing point core are undominated.

Before dynamics are discussed, an important tangent is pursued in section 4. The directional core is found to be equivalent to the cone whose nonexistence is shown by Schofield [1977a] to imply local cycling. Hence the nonexistence of an undominated direction implies that any two points sufficiently close to the status quo can be connected by a finite sequence of points, each of which dominates the preceeding one.

In section 5 an investigation is begun of the paths generated when the status quo is infinitesimally shifted in undominated directions. A status quo so shifted through a point  $x$  is shown, at the time it is at  $x$ , to be simultaneously approaching every point in every winning coalition's preferred-to- $x$  set. No path satisfying this "approach" property exists through points with empty directional cores. Thus a point with an empty directional core satisfies a solution-like property in not being able to shift so as to approach simultaneously every point preferred to itself by every winning coalition.

In section 6 it is shown that if preferences are Euclidean, and if the speed of the status quo is bounded below when it follows undominated directions (which it does whenever they exist, by assumption), then the status quo either enters the set of points with empty directional cores or converges to the point core. This convergence result in the simplest of situations concludes the paper, except for a discussion in section 7 of the principal shortcoming of the directional core as a solution concept: its frequent non-existence.

## 2. THE DIRECTIONAL CORE

The set of possible social states in this paper is simply a Euclidean space  $E^m$ . The societal status quo can therefore be represented at any time by a point  $x \in E^m$ . This section describes the static game that is played at each point in time and whose outcomes are shifts in the status quo.

The magnitude of feasible shifts is assumed to be very small (infinitesimal) and independent of their direction. Hence an outcome may be represented by the direction in which  $x$  shifts, where a direction is formally defined to be a vector in  $E^m$  of unit or zero length. All directions of shift are allowed, so the set of feasible outcomes is  $B = B \cup \{0\}$ , where  $B$  is the ball consisting of all unit vectors.

The players of the game are represented by an index set  $N = \{1, 2, \dots, n\}$ . The preference ordering of each player  $i \in N$  over social states is represented by a continuously differentiable utility function  $u_i: E^m \rightarrow R$ . From  $u_i$ , we induce a preference ordering  $P_i(x)$  on  $B$  by defining, for any  $v_1, v_2 \in B$ ,

$$(2.1) \quad v_1 P_i(x) v_2 \iff v_1 \cdot \nabla u_i(x) > v_2 \cdot \nabla u_i(x).$$

By this ordering on  $B$ , the preferred member of a pair of directions is the one closest to the utility gradient. In Appendix A we show that  $v_1 P_i(x) v_2$  implies that player  $i$  prefers shifting  $x$  infinitesimally in direction  $v_1$  rather than  $v_2$ .

Returning to the game, its outcome (direction  $x$  shifts) shall be determined by the set  $W$  of winning coalitions that characterize a

simple game. Formally,  $W$  is a collection of subsets of  $N$  that is

- (2.2i) (non-trivial)  $\emptyset \notin W, N \in W$
- (2.2ii) (superadditive)  $M \in W, M \subset M' \implies M' \in W$
- (2.2iii) (proper)  $M \in W \implies M^c \notin W$ .

Sometimes we shall assume the game is also strong:

- (2.2iv) (strong)  $M \in W \iff M^c \notin W$ .

Majority rule games, where any coalition containing more than  $n/2$  members is declared winning, are the most common simple games satisfying (2.2i-iii). A majority rule game is strong provided  $n$  is odd.

Now the usual solution concept for a cooperative game, the core, can be defined. First define a dominance relation on  $B$  by  $v_1 D(x) v_2$  provided there exists a winning coalition  $M \in W$  such that  $v_1 P_i(x) v_2$  for all  $i \in M$ . Then the directional core  $K(x)$  is the set of all undominated directions:

$$(2.3) \quad K(x) = \{\bar{v} \in B \mid \nexists v \in B \ni v D(x) \bar{v}\}.$$

If it is nonempty, the outcome shift is assumed to be in  $K(x)$ , which is particularly plausible because  $K(x)$  is shown in Appendix A to contain the core defined there independently of utility gradients.

The nature of the directional core is clarified by the following fundamental characterization. Its statement requires, for any  $v \in B$  and  $x \in E^m$ , a coalition to be defined by

$$(2.4) \quad M(x, v) = \{i \in N \mid v \cdot \nabla u_i(x) > 0\}.$$

$M(x, v)$  is simply the coalition that prefers the status quo to shift in direction  $v$  to remaining at  $x$ , that is, the set of people that prefers (by  $P_i(x)$ )  $v$  to 0.

Proposition 2.1: For any  $x \in E^m$ ,

$$(2.5) \quad K(x) = \{\bar{v} \in B \mid \forall v \in B \ni v \cdot \bar{v} \leq 0, M(x, v) \notin \mathcal{W}\}.$$

Proof: Suppose  $\bar{v} \in B$  and that for all  $v \in B$  satisfying  $v \cdot \bar{v} \leq 0$ ,  $M(x, v) \notin \mathcal{W}$ . If  $\bar{v} \notin K(x)$ , then there exists  $v' \in B$  and an  $M \in \mathcal{W}$  such that  $v' P_i(x) \bar{v}$  for all  $i \in M$ . Hence for each  $i \in M$ ,  $(v' - \bar{v}) \cdot \nabla u_i(x) > 0$ .

Since  $v' - \bar{v} \neq 0$ , we can let  $v = \frac{v' - \bar{v}}{\|v' - \bar{v}\|}$ . Clearly  $v \in B$ , and by

the Cauchy-Schwarz inequality,

$$v \cdot \bar{v} = \frac{v' \cdot \bar{v} - \bar{v} \cdot \bar{v}}{\|v' - \bar{v}\|} \leq 0.$$

So by the hypothesis,  $M(x, v) \notin \mathcal{W}$ . But for any  $i \in M$ ,  $v \cdot \nabla u_i(x) > 0$  since  $(v' - \bar{v}) \cdot \nabla u_i(x) > 0$ . Hence  $M \subset M(x, v)$ , and by superadditivity, we achieve the contradiction  $M(x, v) \in \mathcal{W}$ . Thus  $\bar{v} \in K(x)$ .

Conversely, suppose  $\bar{v} \in K(x)$ . If  $\bar{v} = 0$ , then for any  $v \in B$ ,  $v P_i(x) \bar{v}$  for all  $i \in M(x, v)$ . Because  $\bar{v} = 0$  is undominated,  $M(x, v) \notin \mathcal{W}$  for all  $v \in B$ . Also,  $M(x, 0) = \emptyset \notin \mathcal{W}$ . Hence we need to show  $M(x, v) \notin \mathcal{W}$  only for  $\bar{v} \neq 0$ ,  $v \neq 0$ , and  $v \cdot \bar{v} \leq 0$ .

For any  $i \in N$ ,  $v \cdot \nabla u_i(x)$  can be considered as a continuous function of  $v$  on  $B$ . So if  $v \cdot \nabla u_i(x) > 0$ , there is an open neighborhood  $U_i(v)$  of  $v$  such that  $y \cdot \nabla u_i(x) > 0$  for all  $y \in U_i(v)$ . Hence for any  $v \in B$  and any

$$y \in \bigcap_{i \in M(x, v)} U_i(v) = U(v),$$

$M(x, v) \subset M(x, y)$ . By superadditivity,  $M(x, y) \notin \mathcal{W}$  implies  $M(x, v) \notin \mathcal{W}$ .

If  $v \cdot \bar{v} \leq 0$ , furthermore, since  $U(v)$  is an open neighborhood of  $v$ , there is a  $y \in U(v)$  such that  $y \cdot \bar{v} < 0$ . Therefore, to show that  $v \cdot \bar{v} \leq 0$  implies  $M(x, v) \notin \mathcal{W}$ , we need only show that  $v \cdot \bar{v} < 0$  implies  $M(x, v) \notin \mathcal{W}$ .

So suppose  $v \cdot \bar{v} < 0$ . Let  $v' = \bar{v} - 2(v \cdot \bar{v})v$ . Then  $v' \in B$ .

If  $i \in M(x, v)$ , then  $v \cdot \nabla u_i(x) > 0$ , and so  $(v' - \bar{v}) \cdot \nabla u_i(x) = -2(v \cdot \bar{v})(v \cdot \nabla u_i(x)) > 0$ . Hence  $v' P_i(x) \bar{v}$  for all  $i \in M(x, v)$ , implying that  $M(x, v) \notin \mathcal{W}$  since  $\bar{v}$  is undominated.

The content of proposition 2.1 is easily interpretable. Say that a direction  $v$  "points away" from another direction  $\bar{v}$  provided  $v \cdot \bar{v} \leq 0$ . Then (2.5) implies a direction  $\bar{v}$  is undominated provided no winning coalition prefers a direction pointing away from  $\bar{v}$  over the null direction. Stated differently, if the coalition  $M(x, v)$  preferring  $x$  to shift in direction  $v$  is winning, then no direction  $\bar{v}$  pointing away from  $v$  is undominated.

For future reference, let

$$(2.6) \quad I = \{x \in E^m \mid K(x) \neq \emptyset\},$$

and

$$(2.7) \quad L = \{x \in E^m \mid K(x) = \emptyset\} = I^c.$$

In Appendix A,  $I$  is shown to be closed, so that  $I$  is open. Let  $J = \text{interior } I$  and  $\bar{L} = \text{closure } L$ .

Examples of undominated directions are easily constructed that utilize the benchmark Euclidean preferences so pervasive in the spatial model literature. A person  $i \in N$  is said to have Euclidean preferences if there is a point  $p_i \in E^m$  such that

$$(2.8) \quad u_i(x) = -\|p_i - x\|^2$$

represents them. The point  $p_i$  is  $i$ 's ideal point, and his indifference surfaces are spheres centered at  $p_i$ . At any point  $x$ , the gradient  $\nabla u_i(x) = 2(p_i - x)$  is a vector "pointing" from  $x$  to  $p_i$ .

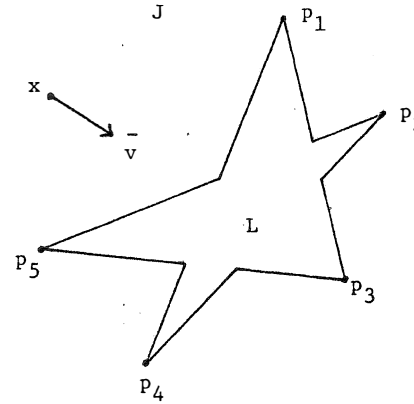
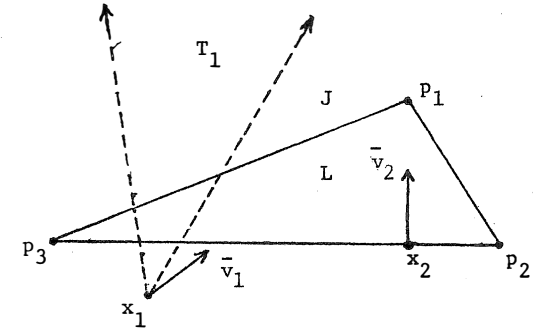
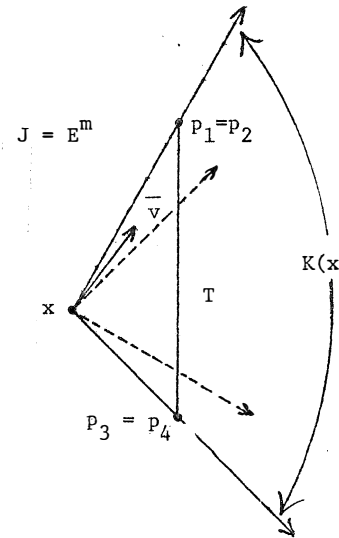
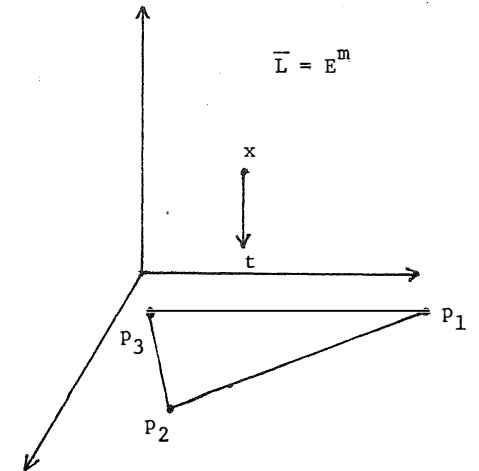
When preferences are Euclidean and the game is majority rule, expression (2.5) simply says that  $\bar{v} \in K(x)$  if any hyperplane containing  $x$  has no more than half the ideal points on any open side of it not containing  $\bar{v}$ . Thus when  $n$  is odd, as in figures 2.1a,b,d, an undominated direction at  $x$  is unique and must point towards a  $p_i$  satisfying the median-like property that any hyperplane containing  $p_i$  and  $x$  bisects the whole set of ideal points. In figure 2.1b the cone  $T_1$  contains the directions that all three people prefer to a null shift at  $x_1$ , but the undominated direction  $\bar{v}_1 \notin T_1$ . At  $x_2$  in figure 2.1b and at  $x$  in figure 2.1c, no winning coalition prefers the undominated direction shown to the zero direction, that is, no winning coalition is better off if those status quos shift in the undominated directions indicated. In figure 2.1d,  $m = 3$  and  $x$  is floating above the two-dimensional triangle  $p_1 p_2 p_3$ . Everybody would prefer  $x$  to shift in a direction such as  $t$ , but nevertheless there is no undominated direction.

### 3. THE POINT CORE

In this section, to further clarify the nature of the directional core  $K(x)$ , it is contrasted to a local point core often considered in the literature. To this end, define  $K \subset E^m$  to be the set of points  $x$  for which there is no direction  $v \in B$  and coalition  $M \in \mathcal{W}$  such that  $v \cdot \nabla u_i(x) > 0$  for all  $i \in M$ . In the previous notation, this point core is simply

$$(3.1) \quad K = \{x \in E^m \mid \forall v \in B, M(x, v) \notin \mathcal{W}\}.$$

FIGURE 2.1

(a)  $m = 2, n = 5$ (b)  $m = 2, n = 3$ (c)  $m = 2, n = 4$ (d)  $m = 3, n = 3$

Although, as is shown in Appendix B,  $K$  is only a linear approximation to the set of locally undominated points, it has been discussed widely under various guises: it is the "local core" to the dynamic game of Schofield [1977b], the set of "Plott equilibriums" in Sloss [1973], and, in the context of majority rule, the set of "quasi-undominated" points in Matthews [1977], of "total medians" in Hoyer and Mayer [1975], and of "equilibriums" in Plott [1967].

The definition of  $K$  can also be written

$$(3.2) \quad K = \{x \in E^m \mid 0 \in K(x)\},$$

which says that  $x$  is in the point core provided no direction in  $B$  dominates the zero direction. This is in contrast to the condition implied by (2.5) for the directional core  $K(x)$  to be non-empty, namely, that only some closed half of  $B$  not dominate the zero direction. In this sense the existence conditions for  $K(x)$  are weaker than those for  $K$ . This is further indicated by the following corollary to proposition 1, which indicates  $x \in K$  if and only if every direction is undominated at  $x$ .

Corollary 3.1: Expression (3.2) can be strengthened to

$$(3.3) \quad K = \{x \in E^m \mid K(x) = B\}.$$

Proof: By (3.2) we need only show that  $K \subset \{x \in E^m \mid K(x) = B\}$ .

Suppose  $x \in K$  and  $\bar{v} \in B$ . By (3.1),  $M(x, v) \notin W$  for all  $v \in B$ .

Hence by (2.5),  $\bar{v} \in K(x)$ . This proves  $K(x) = B$ .

There is a closer relationship between the cores  $K$  and  $K(x)$  in the case of Euclidean preferences. The following proposition states that in this case any direction pointing from  $x$  to  $K$  is contained in the directional core  $K(x)$ -- a result clearly having content only when  $K \neq \emptyset$ .

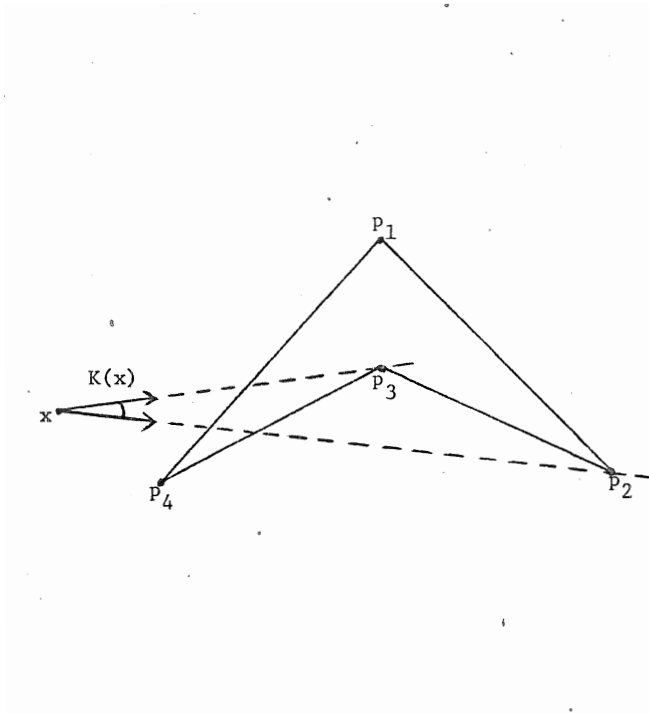
Proposition 3.1: Let  $x \in E^m$ . If preferences are Euclidean, then

$$(3.4) \quad \{\bar{v} \in B \mid \exists \lambda \geq 0 \ni x + \lambda \bar{v} \in K\} \subset K(x).$$

Proof: It must be shown that if  $z \in K$ , then  $K(x)$  contains the  $\bar{v} \in B$  for which  $z = x + \lambda \bar{v}$  for some  $\lambda \geq 0$ . Suppose  $v \in B$  satisfies  $v \cdot \bar{v} \leq 0$ . Then  $v \cdot (z - x) \leq 0$ . Since  $\nabla u_i(x) = 2(p_i - x)$ ,  $v \cdot (p_i - x) > 0$  for all  $i \in M(x, v)$ . So for all  $i \in M(x, v)$ ,  $v \cdot (p_i - z) = v \cdot (p_i - x) - v \cdot (z - x) > 0$ . This proves that  $M(x, v) \subset M(z, v)$ . Since  $M(z, v) \notin W$  because  $z \in K$ , superadditivity implies  $M(x, v) \notin W$ . Thus by proposition 2.1,  $\bar{v} \in K(x)$ .

The reverse of inclusion (3.4) is not always true, as figure 3.1 indicates. In this figure,  $N = \{1, 2, 3, 4\}$  and three- and four-person coalitions are winning. At the point  $x$ ,  $K(x)$  contains all directions between the directions that point to  $p_2$  and  $p_3$ , but only  $p_3$  is contained in  $K$ . The reason that all directions in  $K(x)$  in this example do not point to  $K$  is that the number of players in this majority rule game is even, which means that the game is not strong. The next proposition states that in strong simple games where preferences are Euclidean and  $K \neq \emptyset$ ,  $K(x)$  is exactly the set of directions that point to  $K$ .

Figure 3.1



Proposition 3.2: If preferences are Euclidean, the game is strong, and  $K \neq \emptyset$ , then

$$(3.5) \quad K(x) = \{\bar{v} \in B \mid x + \lambda \bar{v} \in K \text{ for some } \lambda \geq 0\}.$$

Proof: In view of proposition 3.1, it is only necessary to show that  $K(x)$  is contained in the set on the right of (3.5). So let  $\bar{v} \in B$ , and suppose  $x + \lambda \bar{v} \notin K$  for all  $\lambda \geq 0$ . We must show  $\bar{v} \notin K(x)$ . We can assume  $\bar{v} \neq 0$ , for  $\bar{v} = 0$  and  $x \notin K$  imply  $\bar{v} \notin K(x)$ . In Appendix B it is shown that  $K$  is closed, and a simple argument shows it is convex and bounded when preferences are Euclidean. Since  $\{x + \lambda \bar{v} \mid \lambda \geq 0\}$  is disjoint from  $K$ , a separating hyperplane theorem shows the existence of  $v \in B$  such that  $v \cdot \bar{v} < 0$  and  $v \cdot (z - x) > 0$  for any  $z \in K$ . Let  $M = \{i \in N \mid v \cdot (p_i - x) \leq 0\}$ , and let  $z \in K$ . Then if  $i \in M$ ,  $-v \cdot (p_i - z) = v \cdot (z - x) - v \cdot (p_i - x) > 0$ . So, as  $z \in K$ ,  $M \subset M(z, -v) \notin \mathcal{W}$ . Superadditivity now implies  $M \notin \mathcal{W}$ . Since the game is strong and  $M(x, v)$  is the complement of  $M$ ,  $M(x, v) \in \mathcal{W}$ . So by proposition 2.1,  $\bar{v} \notin K(x)$ .

#### 4. LOCAL CYCLES AND DIRECTIONAL CORES

A brief digression is now pursued in order to point out a connection between  $K(x)$  and an important cone studied by Schofield [1977a]. A second characterization of  $K(x)$  is provided that allows an immediate application of Schofield's Null Dual Theorem to show that  $K(x) = \emptyset$  implies the dominance relation over points is cyclic in a neighborhood of  $x$ . Stated differently, a sufficient condition for  $K(x)$  to be nonempty is that local cycling not occur



in the vicinity of  $x$ . More notation is unfortunately necessary.

The (local) Pareto optimal set for a coalition  $M \subset N$  is

$$(4.1) \quad P(M) = \{x \in E^m \mid \nexists v \in B \ni M \subset M(x,v)\},$$

that is,  $x$  is (locally) Pareto optimal for  $M$  if there is no direction in which everyone in  $M$  wants  $x$  to shift. Notice that  $K = \bigcap_{M \in \mathcal{W}} P(M)$ .

The preference co-cone of a coalition  $M \subset N$  at a point  $x$  is simply the convex cone generated by the utility gradients of those in  $M$ :

$$(4.2) \quad \tilde{D}(x,M) = \{y \in E^m \mid y = \sum_{i \in M} \lambda_i \nabla u_i(x), \text{ all } \lambda_i \geq 0, \text{ some } \lambda_i > 0\}.$$

As Schofield [1977a] demonstrates,  $0 \in \tilde{D}(x,M)$  if and only if

$x \in P(M)$ . Define a related cone by

$$(4.3) \quad D(x,M) = \begin{cases} B & \text{if } x \in P(M) \\ \tilde{D}(x,M) \cap B & \text{if } x \notin P(M). \end{cases}$$

Thus  $0 \in D(x,M)$ ,  $D(x,M) = B$ , and  $x \in P(M)$  are all equivalent statements. The next proposition provides an important characterization of the directional core  $K(x)$  in terms of these cones.

Proposition 4.1: At any  $x \in E^m$ ,

$$(4.4) \quad K(x) = \bigcap_{M \in \mathcal{W}} D(x,M).$$

Proof:  $K(x) \subset \bigcap_{M \in \mathcal{W}} D(x,M)$  is first shown. Suppose  $\bar{v} \in K(x)$ . Then we

must show  $\bar{v} \in D(x,M)$  whenever  $M$  is winning, which is nontrivial only when  $D(x,M) \neq B$ . In this case the closed convex cone  $\tilde{D}(x,M)$  does not contain 0. Assume  $\bar{v} \notin D(x,M)$ . Then  $\tilde{D}(x,M)$  and  $\bar{v}$  may be strictly separated with a hyperplane containing the origin, that is, there exists  $v \in B$  such that  $v \cdot \bar{v} \leq 0$  and  $v \cdot y > 0$  for all  $y \in \tilde{D}(x,M)$ . As  $\nabla u_i(x) \in \tilde{D}(x,M)$  for all  $i \in M$ , the latter inequality

implies that  $M \subset M(x,v)$ . Superadditivity then implies  $M(x,v) \in \mathcal{W}$ ,

which by proposition 2.1 contradicts  $\bar{v} \in K(x)$ . Therefore  $\bar{v} \in \tilde{D}(x,M) \cap B = D(x,M)$ .

Now suppose  $\bar{v} \in \bigcap_{M \in \mathcal{W}} D(x,M)$ . We must show  $\bar{v} \in K(x)$ .

Suppose  $v \in B$  satisfies  $v \cdot \bar{v} \leq 0$ . For any  $i \in M(x,v)$ ,  $v \cdot \nabla u_i(x) > 0$ ,

which implies that  $v \cdot y > 0$  for all  $y \in \tilde{D}(x,M(x,v))$ . Hence

$\{0, \bar{v}\} \not\subset \tilde{D}(x,M(x,v))$ . If  $M(x,v) \in \mathcal{W}$ , then by hypothesis,  $\bar{v} \in D(x,M(x,v)) = B \cap \tilde{D}(x,M(x,v))$ , a contradiction. Hence  $M(x,v) \notin \mathcal{W}$ . So by proposition 2.1,  $\bar{v} \in K(x)$ .

Proposition 4.1 allows the immediate conclusion that the emptiness of  $K(x)$  implies local cycling, once the latter is properly defined.

Say that a point  $x_1$  is reachable from a point  $x_0$  provided there is a continuous path  $c: [0,1] \rightarrow E^m$ , differentiable on the intervals

$I_1 = (0, t_1)$ ,  $I_2 = (t_1, t_2)$ , ...,  $I_k = (t_{k-1}, 1)$ , such that

$$(4.5i) \quad c(0) = x_0,$$

$$(4.5ii) \quad c(1) = x_1, \text{ and}$$

$$(4.5iii) \quad M(I_j) = \bigcap_{t \in I_j} M(c(t), c'(t)) \in \mathcal{W} \quad (j = 1, 2, \dots, k).$$

So at each point  $t \in I_j$ , there will be a winning coalition  $M(I_j)$  that prefers the point  $c(t)$  to shift along the curve  $c$  rather than not shift at all.<sup>3</sup>

For any points  $y, z \in E^m$ , say that  $y$  dominates  $z$  if there exists  $M \in \mathcal{W}$  such that  $u_i(y) > u_i(z)$  for all  $i \in M$ . Since

$c'(t) \cdot \nabla u_i(c(t)) > 0$  at each  $t \in I_j$  and  $i \in M(I_j)$ , it is easy to show that  $u_i(c(t_{j+1})) > u_i(c(t_j))$  for every  $i \in M(I_j)$ . Therefore, if  $x_1$  is reachable from  $x_0$ , there is a sequence of  $k + 1$  points  $x_0 = c(0), c(t_1), \dots, c(t_{k-1}), c(1) = x_1$  such that each point dominates the preceding one. Thus if  $x_0 = x_1$ , this dominance relation is cyclic. Local cycling is said to occur at  $x$  provided there is a neighborhood  $U$  of  $x$  such that any point in  $U$  is reachable from  $x$  by a path that stays in  $U$ .<sup>4</sup>

The Null Dual Theorem of Schofield [1977a] states that local cycling occurs at  $x$  if  $\bigcap_{M \in \mathcal{W}} D(x, M)$  is empty. Proposition 4.1 therefore immediately implies

Corollary 4.1: Local cycling occurs at  $x$  if  $K(x) = \emptyset$ .

## 5. THE APPROACH PROPERTY

In this section an examination of dynamics is initiated by characterizing points with nonempty directional cores in terms of certain paths containing them. Specifically, the directional core at  $x$  is nonempty if and only if there is a path through  $x$  that possesses a type of optimality that will soon be defined.

Because the global properties of paths are of interest, utility functions are often subsequently assumed to be pseudo-concave, that is, to satisfy for each  $i \in N$

$$(5.1) \quad (y - x) \cdot \nabla u_i(x) \leq 0 \implies u_i(y) \leq u_i(x).$$

The next proposition will also require the preferred-to- $x$  set of a coalition  $M \subset N$  to be defined by

$$(5.2) \quad P(x, M) = \{y \in E^m \mid u_i(y) > u_i(x) \text{ for all } i \in M\}.$$

The set  $P(x, M)$  is open and, if utility functions are pseudoconcave, also convex.

If  $A$  is either a set or point in  $E^m$ , and  $c: [0, \infty] \rightarrow E^m$  is a continuous, differentiable (almost everywhere) path, let the function  $g_c(\cdot; A): [0, \infty] \rightarrow \mathbb{R}^+$  be the distance from  $c(t)$  to  $A$ :

$$(5.3) \quad g_c(t; A) = \inf_{y \in A} \|y - c(t)\|.$$

Denote by  $g'_c(t; Z)$  the derivative of  $g_c$  at  $t$ . Say that the path  $c$  has the approach property at the point  $c(\bar{t})$  provided that for all  $M \in \mathcal{W}$  and  $y \in P(c(\bar{t}), M)$ ,

$$(5.4) \quad g'_c(\bar{t}; y) < 0.$$

The approach property can be interpreted as a pointwise optimality condition on paths, since a path satisfying the approach property at  $x = c(\bar{t})$  is moving at time  $\bar{t}$  simultaneously towards the preferred-to- $x$  set of every winning coalition. One consequence of the following proposition is that a path satisfying the approach property at a point  $x$  exists if and only if  $K(x) \neq \emptyset$ .

Proposition 5.1: Fix  $x \in E^m$ . If there is a path  $c$  having the approach property at  $x = c(\bar{t})$ , then

$$(5.5) \quad \frac{c'(\bar{t})}{\|c'(\bar{t})\|} \in K(x).$$

Conversely, if each  $u_i$  is pseudoconcave and  $c$  is a path satisfying  $c(\bar{t}) = x$  and (5.5), then  $c$  satisfies the approach property at  $x$ .

Proof: Suppose  $c$  has the approach property at  $x = c(\bar{t})$ . Let

$$\bar{v} = \frac{c'(\bar{t})}{\|c'(\bar{t})\|}. \text{ Suppose } v \in B \text{ satisfies } v \cdot \bar{v} \leq 0. \text{ By the continuity}$$

of each  $u_i$  and the finiteness of  $M(x, v)$ , there exists  $\lambda > 0$  such that  $u_i(x + \lambda v) > u_i(x)$  for all  $i \in M(x, v)$ . Hence, letting  $y = x + \lambda v$ , we have  $y \in P(x, M(x, v))$ . Since

$$g_c'(\bar{t}; y) = -\|c'(\bar{t})\| \quad (v \cdot \bar{v} \geq 0,$$

(5.4) implies that  $M(x, v) \notin \mathcal{W}$ . Proposition 2.1 now implies  $\bar{v} \in K(x)$ , or rather, (5.5).

Conversely, suppose  $c$  is a path satisfying  $c(\bar{t}) = x$  and (5.5), and assume utility functions are pseudoconcave. Let  $y \in P(x, M)$  for some  $M \in \mathcal{W}$ . Then by pseudoconcavity,  $(y - x) \cdot \nabla u_i(x) > 0$  for each  $i \in M$ . Thus, by (5.5) and proposition 2.1,  $M \in \mathcal{W}$  implies  $(y - x) \cdot c'(\bar{t}) > 0$ . Hence  $c$  satisfies the approach property at  $x$ :

$$g_c(\bar{t}; y) = \frac{-c'(\bar{t}) \cdot (y - x)}{g_c(\bar{t}; y)} < 0.$$

Proposition 5.1 confers the optimal-like approach property to paths that always travel in undominated directions when such exist, as will be explicitly stated in the next section. Furthermore, proposition 5.1 confers a solution property of sorts to the set  $L$ , since a point contained in  $L$  has an empty directional core and therefore cannot simultaneously approach every point preferred to

itself by every winning coalition.<sup>5</sup> Therefore points in either  $K$  or  $L$  satisfy desirable properties; points  $x \in K$  strongly because the preferred-to- $x$  set of every winning coalition is empty, and points  $x \in L$  weakly because they cannot simultaneously approach all winning coalitions' preferred-to- $x$  sets.

Again, stronger results are obtainable if preferences are Euclidean. This section concludes with the following results that will be important for the convergence theorem of the next section.

Lemma 5.1: Suppose preferences are Euclidean, and assume  $\bar{x} \notin P(M)$  for some  $M \subset N$ . Let  $\bar{z} \in P(M)$  satisfy

$$\|\bar{z} - \bar{x}\| = \inf_{z \in P(M)} \|z - \bar{x}\|.$$

Then  $\bar{z} \in P(\bar{x}, M)$ .

Proof: It is well-known that  $P(M)$  is the convex hull of  $\{p_i \mid i \in M\}$ . Hence  $\bar{z}$  exists, since  $P(M)$  is closed. As  $P(M)$  is also convex, there is a supporting hyperplane at  $\bar{z}$  with normal  $(\bar{z} - \bar{x})$ , that is,

$$(5.6) \quad (z - \bar{z}) \cdot (\bar{z} - \bar{x}) \geq 0$$

for all  $z \in P(M)$ . Since each  $p_i \in P(M)$ , let  $z = p_i$  in (5.6), subtract  $\bar{x} \cdot (\bar{z} - \bar{x})$  from both sides, and rearrange to yield

$$(p_i - \bar{x}) \cdot (\bar{z} - \bar{x}) \geq (\bar{z} - \bar{x}) \cdot (\bar{z} - \bar{x}).$$

Hence

$$\begin{aligned} \|p_i - \bar{x}\|^2 &= (p_i - \bar{x}) \cdot (p_i - \bar{x}) \\ &> (p_i - \bar{x}) \cdot (p_i - \bar{x}) - 2(p_i - \bar{x}) \cdot (\bar{z} - \bar{x}) + (\bar{z} - \bar{x}) \cdot (\bar{z} - \bar{x}) \\ &= \|(p_i - \bar{x}) - (\bar{z} - \bar{x})\|^2 \\ &= \|p_i - \bar{z}\|^2. \end{aligned}$$

As preferences are Euclidean, this proves  $u_i(\bar{z}) > u_i(\bar{x})$  for each  $i \in M$ , or rather,  $\bar{z} \in P(\bar{x}, M)$ .

Using this lemma, the following corollary proves that any path through  $\bar{x}$  approaches each  $P(M)$  if and only if its tangent vector at  $\bar{x}$  is contained in  $K(\bar{x})$ . While this property does not by itself have an optimal interpretation like the approach property, it will provide the cornerstone of the next section's convergence result.

Corollary 5.1: Suppose preferences are Euclidean, and fix  $\bar{x} \in E^m$ .

If  $c$  is a path differentiable at  $c(\bar{t}) = \bar{x}$  such that

$$(5.7) \quad \frac{c'(\bar{t})}{\|c'(\bar{t})\|} \in K(\bar{x}),$$

then for all  $M \in \mathcal{W}$  such that  $\bar{x} \notin P(M)$ ,

$$(5.8) \quad g_c'(\bar{t}; P(M)) < 0.$$

Proof: Suppose  $M \in \mathcal{W}$  and  $\bar{x} \notin P(M)$ . Define  $z(t) \in P(M)$  by

$$\|z(t) - c(t)\| = \inf_{z \in P(M)} \|z - c(t)\| = g_c(t; P(M)).$$

Since  $c(t)$  is continuous and  $P(M)$  convex,  $z(t)$  is continuous. We first show  $\phi(t) = z(t) \cdot (\bar{z} - \bar{x})$  is differentiable with  $\phi'(\bar{t}) = 0$ , where we have let  $\bar{z} = z(\bar{t})$ .

Because  $P(M)$  is convex, there is a supporting hyperplane at  $z(t)$  with normal  $z(t) - c(t)$ :

$$(5.9) \quad (z - z(t)) \cdot (z(t) - c(t)) \geq 0$$

for all  $z \in P(M)$ . Hence

$$\liminf_{t \rightarrow \bar{t}+} \left[ \frac{(z(t) - \bar{z}) \cdot (\bar{z} - \bar{x})}{t - \bar{t}} \right] \geq 0,$$

and

$$\begin{aligned} \limsup_{t \rightarrow \bar{t}+} \left[ \frac{(z(t) - \bar{z}) \cdot (\bar{z} - \bar{x})}{t - \bar{t}} \right] \\ = \limsup_{t \rightarrow \bar{t}+} \left[ \frac{(z(t) - \bar{z}) \cdot (z(t) - c(t))}{t - \bar{t}} \right] \leq 0. \end{aligned}$$

Hence the right hand derivative at  $\bar{t}$  of  $\phi(t)$ , equal to

$$\lim_{t \rightarrow \bar{t}+} \left[ \frac{(z(t) - \bar{z}) \cdot (\bar{z} - \bar{x})}{t - \bar{t}} \right],$$

exists and is 0. A similar argument establishes the same for the left hand derivative, so that  $\phi'(\bar{t})$  exists and  $\phi'(\bar{t}) = 0$ .

Since  $\phi'(\bar{t})$  and  $c'(\bar{t})$  exist,

$$\begin{aligned} g_c'(\bar{t}; P(M)) &= \frac{d[(z(t) - c(t)) \cdot (\bar{z} - \bar{x})]}{dt} \Big|_{t=\bar{t}} g_c(\bar{t}; P(M))^{-1} \\ &= [\phi'(\bar{t}) - c'(\bar{t}) \cdot (\bar{z} - \bar{x})] g_c(\bar{t}; \bar{z})^{-1} \end{aligned}$$

also exists. As  $\phi'(\bar{t}) = 0$ ,

$$\begin{aligned} (5.10) \quad g_c'(\bar{t}; P(M)) &= - [c'(\bar{t}) \cdot (\bar{z} - \bar{x})] g_c(\bar{t}; \bar{z})^{-1} \\ &= g_c'(\bar{t}; \bar{z}). \end{aligned}$$

But by lemma 5.1,  $\bar{z} \in P(\bar{x}, M)$ . Hence if (5.7) is true, proposition 5.1 implies  $g_c'(\bar{t}; \bar{z}) < 0$  since Euclidean utility functions are pseudo-concave. Thus (5.7) implies (5.8).

## 6. THE DYNAMIC PROCESS

Now consider paths that the status quo traces if at each time its direction of shift is contained in the directional core whenever it is nonempty. The requirement that a direction of movement be undominated whenever possible is a behavioral restriction. These paths are generated when the outcome of the simple game is an infinitesimal shift of the status quo, after which a new game is played at the new status quo, and the entire process repeated indefinitely. One key assumption here is that players do not respond to realizations that current actions determine the location of future status quos and hence which games will subsequently be played. Whether this "sincere" behavior is a result of myopia, moral injunctions against large-scale gaming, etc., it probably occurs in many situations.

This dynamic process is modeled here as simply as possible. The ultimate goal is to obtain convergence of some sort to the set  $K \cup L$  that was argued previously to satisfy solution-like properties. However, for mathematical convenience, convergence to the closed set  $K \cup \bar{L}$  is investigated. The simplest assumption sufficient for convergence is merely that the speed of the status quo  $x$  is bounded below by  $s > 0$  when  $x$  is not in  $K \cup \bar{L}$ , that is, when  $x \in J \setminus K$ . An upper bound  $S$  on the speed is also a convenient assumption. Finally, in order to minimally restrict the direction of motion, it is required to be undominated only when  $x \in J \setminus K$  rather than when  $x \in I$ . Summarizing, the status quo is assumed to follow a

path  $x: [0, \infty] \rightarrow E^m$ , differentiable almost everywhere, satisfying

$$(6.1) \quad x'(t) \in F(x(t)),$$

where  $F: E^m \rightarrow 2^{E^m}$  is a correspondence defined by

$$(6.2) \quad F(x) = \begin{cases} \{y \in E^m \mid \|y\| \leq S\} & \text{if } x \in K \cup \bar{L} \\ \{y \in E^m \mid s \leq \|y\| \leq S \text{ and } \frac{y}{\|y\|} \in K(x)\} & \text{if } x \in J \setminus K. \end{cases}$$

The correspondence  $F$  maps points into truncated, convex closed cones, and is shown to be uppersemicontinuous in Appendix C.

It now immediately follows that such a path almost always satisfies the approach property whenever possible.

Corollary 6.1: Provided all preferences are pseudoconcave, a path  $x$  satisfying (6.1) and (6.2) has the approach property at all  $x(t) \in J$ .

Proof: If  $x(t) \in J \setminus K$ , then from (6.2),  $\frac{x'(t)}{\|x'(t)\|} \in K(x)$ . Hence proposition 5.1 immediately implies that  $x$  satisfies the approach property at  $x(t)$ . If  $x(t) \in K$ , then the approach property is vacuously satisfied at  $x(t)$  since  $P(x(t), M) = \emptyset$  for each  $M \in \mathcal{W}$ .

Two types of convergence will be discussed now. If  $c$  is a path in  $E^m$  and  $A \subset E^m$ ,  $c$  is said to converge to  $A$  provided  $\lim_{t \rightarrow \infty} g_c(t; A) = 0$ . The path  $c$  is said to enter  $A$  provided that given any  $T \geq 0$ , there is a time  $t \geq T$  such that  $c(t) \in A$ .

The next proposition is that an  $x(t)$  satisfying (6.1) and (6.2) will converge to  $K$  if  $K \neq \emptyset$  or will enter  $\bar{L}$  if  $K = \emptyset$ , provided that preferences are Euclidean. Hence in this case the path converges

to the set  $K \cup \bar{L}$  that was argued to have solution properties in the previous section. From corollary 5.1 we see that  $x(t)$  will move into one Pareto set  $P(M)$  after another, never leaving any after entering, as long as  $x(t) \in J$ . So what occurs is that  $x(t)$  keeps moving simultaneously towards all winning coalitions' Pareto sets that do not contain it until it has either moved into them all ( $x \in K$ ) or can no longer approach them all simultaneously ( $x \in \bar{L}$ ).

**Proposition 6.1:** Suppose all preferences are Euclidean. If  $K \neq \emptyset$ , then an  $x(t)$  satisfying (6.1) and (6.2) converges to  $K$ , and does so monotonically if the game is strong. If  $K = \emptyset$ , then  $x(t)$  enters  $\bar{L}$ .

**Proof:** Suppose first that  $K \neq \emptyset$ . Then proposition 3.1 implies  $J = E^m$ , so that so that  $x(t) \in J$  always. Let  $M \in \mathcal{W}$ . Corollary 5.1 now implies that  $g_x(t; P(M))$  is strictly decreasing in  $t$  when  $x(t) \notin P(M)$ . As  $g_x(t; P(M))$  is bounded below by 0,

$$(6.3) \quad d^* = \lim_{t \rightarrow \infty} g_x(t; P(M))$$

exists. It was shown in proving corollary 5.1 that  $g_x(t; P(M))$  was differentiable when  $x(t)$  was differentiable, which is almost everywhere. Hence (6.3) implies

$$(6.4) \quad \lim_{t \rightarrow \infty} g'_x(t; P(M)) = 0.$$

Since  $x(t)$  is always approaching the compact set  $P(M)$ , the range of the path  $x$  is contained in a compact set. As the range of  $x'$  is also contained in a compact set, there is a sequence  $t_v \rightarrow \infty$  as  $v \rightarrow \infty$

such that  $\bar{x} = \lim_{v \rightarrow \infty} x(t_v)$  and  $\bar{x}' = \lim_{v \rightarrow \infty} x'(t_v)$  exist. Let  $z(t) \in P(M)$  satisfy

$$\|z(t) - x(t)\| = g_x(t; P(M)),$$

and let  $\bar{z} = \lim_{v \rightarrow \infty} z(t_v)$ . Then

$$(6.5) \quad \begin{aligned} \bar{x}' \cdot (\bar{z} - \bar{x}) &= -\lim_{v \rightarrow \infty} g_x(t_v; z(t_v)) g'_x(t_v; z(t_v)) && \text{(by 5.10))} \\ &= -\lim_{v \rightarrow \infty} g_x(t_v; P(M)) g'_x(t_v; P(M)) \\ &= -d^* = 0 && \text{(by (6.3) and 6.4)).} \end{aligned}$$

Since  $F$  is uppersemicontinuous,  $\bar{x}' \in F(\bar{x})$ . If  $d^* \neq 0$ , then  $\bar{x} \notin P(M)$  and so  $\bar{x} \notin K$ . Hence by (6.1) and (6.2),  $\|\bar{x}'\| \geq s > 0$  and

$$(6.6) \quad \frac{\bar{x}'}{\|\bar{x}'\|} \in K(\bar{x}).$$

Let  $c(t)$  be a path such that  $c(\bar{t}) = \bar{x}$  and  $c'(\bar{t}) = \bar{x}'$ . Then, as  $\|\bar{z} - \bar{x}\| = g_c(\bar{t}; P(M))$ , (6.6), corollary 5.1, and  $g_c(\bar{t}; P(M)) > 0$  imply the contradiction

$$\begin{aligned} \bar{x}' \cdot (\bar{z} - \bar{x}) &= -g_c(\bar{t}; \bar{z}) g'_c(\bar{t}; \bar{z}) && \text{(by (5.10))} \\ &= -g_c(\bar{t}; P(M)) g'_c(\bar{t}; P(M)) \\ &> 0. \end{aligned}$$

This proves that  $d^* = 0$ . Hence  $x(t)$  converges to  $P(M)$  for each  $M \in \mathcal{W}$ , which means that  $x(t)$  converges to  $K = \bigcap_{M \in \mathcal{W}} P(M)$ .

If  $K \neq \emptyset$  and the game is strong, proposition 3.2 implies that  $x'(t)$  always "points" at  $K$ . This can be used to show  $g'(t; K) < 0$  when  $x(t) \notin K$ , so that  $x(t)$  monotonically converges to  $K$ .

Now suppose  $K = \emptyset$ . If there was a  $T$  such that  $x(t) \in J$  for all  $t \geq T$ , the above argument establishes the existence of a limit point  $\bar{x}$  such that  $\bar{x} \in P(M)$  for all  $M \in \mathcal{W}$ . But then  $\bar{x} \in K$ , which is impossible. Hence  $\bar{L}$  exists and  $x(t)$  enters  $\bar{L}$  when  $K = \emptyset$ .

The proof of proposition 6.1 could have used more of the special structure of Euclidean preferences, that is, it could have first been shown via proposition 4.1 that undominated directions "point" towards all winning coalitions' Pareto sets, which indicates that  $x(t)$  must converge to them all if  $x(t) \in J$  always. However, the above proof used the Euclidean assumption only via the monotonicity property of corollary 5.1. This should allow some elements of the proof to be useful in proving convergence to  $K \cup \bar{L}$  under a less restrictive preference assumption.

## 7. THE EXISTENCE PROBLEM

The value of the hypothesis that game outcomes will be undominated is its use as a predictor. In situations where social change is slow, so that the status quo can never shift far, it can be predicted to shift in undominated directions - provided they exist. But unfortunately, directional cores, like other cores, frequently don't exist.

When the dimension  $m$  of the space is large relative to the number of people  $n$ , the problem is severe. Specifically, when  $\mathcal{W}$  consists of coalitions of size  $q < n$  or larger, and  $m \geq \max \{2q - 1, q + 1\}$ , then Schofield [1977a] shows that for almost all  $n$ -tuples of utility functions, the set of points satisfying his Null Dual condition is dense in  $E^m$ . Since it was shown in section 4 that a point satisfying his Null Dual condition has a null directional core, it must be that "usually" in these games  $\bar{L} = E^m$  and  $J = \emptyset$ . Not only does this make the convergence result of section 6 trivially true, but it makes the existence of undominated directions so rare it seems useless to study them.

However, often the number of individuals in the game is far larger than the dimensionality of the space, such as when the amounts of a few public goods must be decided upon in a large society. In these cases  $J$  is often sizeable, as the examples in figures 2.1a-c illustrate. Nevertheless, proposition 2.1 implies that a necessary condition for  $K(x) \neq \emptyset$  in the case of majority rule is that at least half the utility gradients lie on any closed side of any hyperplane containing  $x$  and  $x + \bar{v}$ , where  $\bar{v} \in K(x)$ . This condition of symmetry about the line determined by  $x$  and  $x + \bar{v}$  is only slightly less restrictive, if  $m > 2$ , than Plott's [1967] symmetry condition about the point  $x$  necessary for  $x \in K$ .

In fact, if there is a fraction  $\lambda$  such that

$$\mathcal{W} = \{M \subset N \mid \lambda n \leq |M|\},$$

then the set of utility gradients satisfies a pairwise symmetry

condition if  $\bar{v} \in K(x)$ . Observe from proposition 2.1 that  $\bar{v} \in K(x)$  if and only if  $x$  would be in the point core  $K$  if the only feasible directions in which  $x$  could shift were in

$$D = \{v \in B \mid v \cdot \bar{v} \leq 0\}.$$

The conditions for  $x \in K$  when the set of feasible shift directions is  $D$  is one of the special cases considered in Matthews [1977], and, if  $\lambda = .5$  and  $n$  is odd, the constrained majority rule case considered by Plott [1967].

To apply Matthews [1977], let  $T$  be a two dimensional subspace of  $E^m$  containing  $\bar{v} \in K(x)$ , and let  $N_T = \{i \in N \mid \nabla u_i(x) \in T\}$ . Let  $Q \subset N_T$  be a maximal subset of  $N_T$  that can be partitioned into pairs  $\{i, j\}$  for which neither  $\nabla u_i(x)$  nor  $\nabla u_j(x)$  is a multiple of  $\bar{v}$ , but there is an  $\alpha_i > 0$  and  $\alpha_j > 0$  such that

$$(7.1) \quad \alpha_i \nabla u_i(x) + \alpha_j \nabla u_j(x) \in \{0, \bar{v}\}.$$

Finally, let  $R \subset N_T \setminus Q$  be defined by  $R = \{i \in N \mid \nabla u_i(x) = \alpha \bar{v} \text{ for some } \alpha \geq 0\}$ . Then a result in Matthews [1977] implies that a necessary condition for  $\bar{v} \in K(x)$  is a bound on  $|Q|$ :

$$(7.2) \quad |N_T| - |R| \geq |Q| > |N_T| - 2|R| - (2\lambda - 1)n.$$

To interpret this, suppose the game is majority rule with  $n$  odd. Then  $\lambda = .5$  and we have

$$(7.3) \quad |N_T| - |R| \geq |Q| > |N_T| - 2|R|.$$

If  $|R| = 0$ , then  $Q = N_T$  so that  $N_T$  is even, which cannot be true for all two dimensional subspaces  $T$  because  $n$  is odd. Hence (7.3) implies  $|R| \geq 1$ . If  $|R| = 1$ , then (7.3) implies  $|Q| = |N_T| - 1$ . In this case, by applying (7.3) to all two dimensional subspaces containing

$\bar{v}$ , we see that the  $n-1$  people in  $N \setminus R$  can be partitioned into pairs whose gradients satisfy (7.1). This is exactly the condition obtained by Plott [1967] for constrained majority rule, and is obviously a restrictive symmetry condition.

Thus we are left with the observation that directional cores, like the cores to most voting games, are often empty. The properties of paths generated by simple games that have been established should still be useful in situations where directional cores usually exist. But a different dynamic hypothesis, not based in any way on an equilibrium or core solution concept, must be formulated to investigate situations where directional cores are empty.



## APPENDIX A

This appendix to section 2 first investigates the relationship between  $K(x)$  and a directional core  $\hat{K}(x)$  defined by a more complete inducement of preferences upon  $B$  than is represented by  $P_i(x)$ . It concludes with a proof that  $I$  is closed.

The best way to induce preferences from  $E^m$  to  $B$  that is in keeping with the spirit of the model is to define a preference ordering  $\hat{P}_i(x)$  on  $B$  by

(A.1)  $v_1 \hat{P}_i(x) v_2 \iff \exists \bar{\lambda} > 0 \ni u_i(x + \lambda v_1) > u_i(x + \lambda v_2) \quad \forall 0 < \lambda < \bar{\lambda}$ .

Player  $i$  will prefer shifting  $x$  in direction  $v_1$  to shifting it in direction  $v_2$ , when both shifts are very small and of equal magnitude, if  $v_1 \hat{P}_i(x) v_2$ . As  $u_i$  is continuously differentiable,  $v_1 \hat{P}_i(x) v_2$  and  $v_2 P_i(x) v_1$  imply  $u_i(x + \lambda v_1) = u_i(x + \lambda v_2)$  for all  $\lambda \geq 0$  less than some  $\bar{\lambda} > 0$ . Thus an indifference relation defined from  $\hat{P}_i(x)$  truly indicates that a player is indifferent between small shifts. This is not true of an indifference relation defined from  $P_i(x)$ , since there are cases where  $v_1 \hat{P}_i(x) v_2$  but not  $v_1 P_i(x) v_2$ .

The ordering  $P_i(x)$  is a linear approximation to  $\hat{P}_i(x)$  and is seen in lemma A1 below to be contained in  $\hat{P}_i(x)$ . The condition for  $P_i(x) = \hat{P}_i(x)$  on  $B = B \setminus \{0\}$  is that  $u_i$  be locally symmetric (about its gradient) at  $x$ , which is defined to mean that for any  $v_1, v_2 \in B$ , there exists  $\bar{\lambda} > 0$  such that

(A.2)  $(v_1 - v_2) \cdot \nabla u_i(x) \leq 0 \implies u_i(x + \lambda v_1) \leq u_i(x + \lambda v_2)$

for all  $0 < \lambda < \bar{\lambda}$ . The name of this property results from the fact that (A.2) is satisfied provided that whenever  $v_1, v_2 \in B$  are equi-

distant from the gradient  $\nabla u_i(x)$ ,  $\lambda v_1$  and  $\lambda v_2$  must be on the same indifference curve for small  $\lambda > 0$ . Euclidean functions as defined in (2.8) and linear functions are two examples of functions everywhere locally symmetric.

Lemma A1:  $P_i(x) \subset \hat{P}_i(x)$ .  $P_i(x) = \hat{P}_i(x)$  on  $B$  if and only if  $u_i$  is locally symmetric at  $x$ .

Proof: If  $v_1 P_i(x) v_2$ , then  $(v_1 - v_2) \cdot \nabla u_i(x) > 0$ . Let  $f(\lambda) = u_i(x + \lambda v_1) - u_i(x + \lambda v_2)$ , and observe that

$$\lim_{\lambda \rightarrow 0+} \frac{f(\lambda)}{\lambda} = f'(0) = (v_1 - v_2) \cdot \nabla u_i(x) > 0.$$

Hence for small  $\lambda > 0$ ,  $f(\lambda) > 0$ , that is,  $v_1 \hat{P}_i(x) v_2$ .

Now suppose that  $\hat{P}_i(x) = P_i(x)$  and that  $(v_1 - v_2) \cdot \nabla u_i(x) \leq 0$  for a particular  $v_1, v_2 \in B$ . Then not  $v_1 P_i(x) v_2$ , and hence not  $v_1 \hat{P}_i(x) v_2$ . Thus there is no  $\bar{\lambda} > 0$  such that  $f(\lambda) > 0$  for all  $0 < \lambda < \bar{\lambda}$ . Since  $f$  is continuously differentiable, this implies the existence of  $\bar{\lambda} > 0$  such that  $f(\lambda) \leq 0$  for all  $0 < \lambda < \bar{\lambda}$ . This proves that  $u_i$  is locally symmetric at  $x$ .

Conversely, suppose  $u_i$  is locally symmetric at  $x$ , and that  $v_1 \hat{P}_i(x) v_2$ . Then  $f(\lambda) > 0$  for all small  $\lambda > 0$ . If  $(v_1 - v_2) \cdot \nabla u_i(x) \leq 0$ , then by (A.2), there exists  $\bar{\lambda} > 0$  such that  $f(\lambda) \leq 0$  for all  $0 < \lambda < \bar{\lambda}$ . As this is impossible,  $(v_1 - v_2) \cdot \nabla u_i(x) > 0$ , implying  $v_1 P_i(x) v_2$ .

If  $v_1, v_2 \in B$ , say that  $v_1 \hat{P}(x) v_2$   $\hat{P}(x)$ -dominates  $v_2$  provided

$M \in \mathcal{W}$  exists such that  $v_1 \hat{P}_1(x) v_2 (v_1 P_1(x) v_2)$  for all  $i \in M$ . Then  $K(x)$  is the set of  $P(x)$ -undominated directions, and similarly define the core  $\hat{K}(x)$  to be the set of  $\hat{P}(x)$ -undominated directions.  $K(x)$  is a linear approximation to  $\hat{K}(x)$ , and the following proposition specifies their relationship. Without further assumptions it implies that the propositions of this paper characterizing directions in  $K(x)$  are also true for directions in  $\hat{K}(x)$ .

Proposition A1:  $\hat{K}(x) \subset K(x)$ . Conversely,  $\hat{K}(x) = K(x)$  provided

either (i) every  $u_i$  is linear, or (ii) every  $u_i$  is locally symmetric at  $x$  and, for each  $M \in \mathcal{W}$ ,  $x \notin P(M)$  or

$x \in \text{interior}\{y \mid \exists z \ni u_i(z) > u_i(y) \forall i \in M\} \equiv \text{IP}(M)$ .<sup>10</sup>

Proof: Lemma A1 implies  $\hat{K}(x) \subset K(x)$ . If each  $u_i$  is linear, then

$\hat{P}_1(x) = P_1(x) \Rightarrow \hat{K}(x) = K(x)$ . Now suppose each  $u_i$  is locally symmetric at  $x$ , and let  $\bar{v} \in K(x)$ . Case 1:  $\bar{v} \neq 0$ . If  $\bar{v} \notin \hat{K}(x)$ , then  $\exists M \in \mathcal{W} \ni 0 \hat{P}_1(x) \bar{v} \forall i \in M$ , as by lemma A1  $\nexists v \neq 0$  that  $\hat{P}(x)$ -dominates  $\bar{v}$ . We know  $x \in P(M)$ , for otherwise

$\exists v \neq 0 \ni v P_1(x) 0 \forall i \in M$ , which by lemma A1 and transitivity implies the contradiction  $v P_1(x) v \forall i \in M$ . Also,  $0 \hat{P}_1(x) \bar{v}$  implies  $\exists \bar{\lambda} > 0 \ni \forall 0 < \lambda < \bar{\lambda}, u_i(x) > u_i(x + \lambda \bar{v}) \forall i \in M$ .

Hence  $x \notin \text{IP}(M)$ . This final contradiction proves  $\bar{v} \in \hat{K}(x)$ .

Case 2:  $\bar{v} = 0$ . Then by (3.1),  $x \in P(M) \forall M \in \mathcal{W}$ , so that  $x \in \text{IP}(M) \forall M \in \mathcal{W}$ . But then no  $v \in B$   $\hat{P}(x)$ -dominates  $\bar{v} = 0$ , so that  $\bar{v} \in \hat{K}(x)$ .

Proposition A2:  $I$  is closed.

Proof: Let  $\{x_t\}$  be a sequence of points in  $I$  converging to  $\bar{x}$ . Let  $v_t \in K(x_t)$ . Since  $B$  is compact we can choose a subsequence  $\{v_k\}$  of  $\{v_t\}$  such that  $\lim v_k = \bar{v} \in B$ . We show that  $\bar{v} \in K(\bar{x})$  and hence  $\bar{x} \in I$ .

If  $\bar{v} = 0$ , then as  $0 \in B$  is an isolated point of  $B$ , there exist  $k_0$  such that  $v_k = 0$  for all  $k \geq k_0$ . Suppose  $i \in M(\bar{x}, v)$  for some  $v \in B$ . Then  $v \cdot \nabla u_i(\bar{x}) > 0$ . As  $\nabla u_i$  is continuous, there exists  $k(i)$  such that  $v \cdot \nabla u_i(x_k) > 0$  for all  $k \geq k(i)$ . Hence  $M(\bar{x}, v) \subset M(x_k, v)$  for all  $k \geq \bar{k} = \max_{i \in M(\bar{x}, v)} \{k_0, k(i)\}$ . Since  $v_k = 0 \in K(x_k)$  for  $k \geq \bar{k}$ , proposition 2.1 implies  $M(x_k, v) \notin \mathcal{W}$  for  $k \geq \bar{k}$ . Superadditivity now implies  $M(\bar{x}, v) \notin \mathcal{W}$ , and proposition 2.1 now implies  $v \in K(\bar{x})$ .

So assume  $\bar{v} \neq 0$ , and suppose  $v \cdot \bar{v} \leq 0$  for some  $v \in B$ . As in the proof of proposition 2.1, the finiteness of  $N$  can be used to show existence of a  $y \in B$  near  $v$  such that  $y \cdot \bar{v} < 0$  and  $M(\bar{x}, v) \subset M(\bar{x}, y)$ . As in the previous paragraph, the continuity of  $\nabla u_i$  implies the existence of  $\bar{k}$  such that  $M(\bar{x}, y) \subset M(x_k, y)$  for all  $k \geq \bar{k}$ . Furthermore, since  $v_k \rightarrow \bar{v}$ , there exists  $\hat{k}$  such that  $y \cdot v_k < 0$  for all  $k \geq \hat{k}$ . If  $M(\bar{x}, v) \in \mathcal{W}$ , then  $M(x_k, y) \in \mathcal{W}$  for all  $k \geq \max\{\bar{k}, \hat{k}\}$ , which implies  $v_k \notin K(x_k)$  for  $k \geq \max\{\bar{k}, \hat{k}\}$  by proposition 2.1. This contradiction shows  $M(\bar{x}, v) \notin \mathcal{W}$  for any  $v$  such that  $v \cdot \bar{v} \leq 0$ . Proposition 2.1 now implies  $\bar{v} \in K(\bar{x})$ .

The purpose of this appendix is analagous to that of appendix A, namely, to examine the relationship between the local point core  $K$  and a truly local point core  $\hat{K}$  defined here. Just as  $K(x)$  was viewed as a linear approximation to  $\hat{K}(x)$ ,  $K$  will be considered to linearly approximate  $\hat{K}$ .

Say that  $x \in E^m$  is locally undominated provided a neighborhood  $U$  of  $x$  exists such that for any  $z \in U$ ,  $\{i \in N \mid u_i(z) > u_i(x)\} \notin \mathcal{W}$ . The local point core  $\hat{K}$  is the set of locally undominated points in  $E^m$ . Say that a function  $u_i$  is locally pseudoconcave at  $x$  provided there exists a radius  $\lambda_i > 0$  such that for any  $v \in B$ ,

$$(B.1) \quad v \cdot \nabla u_i(x) \leq 0 \implies u_i(x + \lambda v) \leq u_i(x)$$

for all  $0 < \lambda < \lambda_i$ . (Observe that local pseudoconcavity is equivalent to pseudoconcavity (see (5.1)) if  $\lambda_i = \infty$ .)

The following lemma, stronger than necessary for proposition B1, is of independent interest because it shows when  $\hat{K}$  can be defined in terms of  $\hat{K}(x)$  just as  $K$  is defined in (3.2) in terms of  $K(x)$ .

Lemma B1:

$$(B.2) \quad \hat{K} \subset \{x \in E^m \mid 0 \in \hat{K}(x)\}.$$

If each  $u_i$  is locally pseudoconcave at each  $x$  contained in the right hand side of (B.2), then

$$(B.3) \quad \hat{K} = \{x \in E^m \mid 0 \in \hat{K}(x)\}.$$

Proof: Suppose  $x \in \hat{K}$ . If  $0 \notin \hat{K}(x)$ , then there exists  $M \in \mathcal{W}$  and  $v \in B$  such that for each  $i \in M$ ,  $v \cdot \nabla u_i(x) > 0$ . Hence for each  $i \in M$ , there

is a  $\bar{\lambda}_i > 0$  such that  $u_i(x + \lambda v) > u_i(x)$  for all  $0 < \lambda < \bar{\lambda}_i$ . As  $M$  is finite,  $\bar{\lambda} = \min_{i \in M} \{\bar{\lambda}_i\} > 0$  and for all  $i \in M$ ,  $u_i(x + \lambda v) > u_i(x)$  if  $0 < \lambda < \bar{\lambda}$ . But now any neighborhood of  $x$  contains a point  $x + \lambda v$  that dominates  $x$  via  $M$ , which contradicts  $x \in \hat{K}$ . Hence  $0 \in \hat{K}(x)$ .

Conversely, suppose  $0 \in \hat{K}(x)$  and each  $u_i$  is locally pseudoconcave at  $x$  with a radius of  $\lambda_i > 0$ . As  $N$  is finite,  $\bar{\lambda} = \min_{i \in N} \{\lambda_i\} > 0$ .

If  $x \notin \hat{K}$ , there exists  $v \in B$  and  $\lambda > 0$  such that  $\lambda < \bar{\lambda}$  and  $u_i(x + \lambda v) > u_i(x)$  for  $i$  contained in some  $M \in \mathcal{W}$ . Hence by (B.1),  $v \cdot \nabla u_i(x) > 0$  for all  $i \in M$ , i.e.,  $v \cdot \nabla u_i(x) > 0$  for all  $i \in M$ . By lemma A1,  $v \cdot \nabla u_i(x) > 0$  for all  $i \in M$ , which contradicts  $0 \in \hat{K}(x)$ . Hence  $x \in \hat{K}$ .

Proposition B1:  $\hat{K} \subset K$ . If each  $u_i$  is locally pseudoconcave at each  $x \in K$ , then  $\hat{K} = K$ .<sup>6</sup>

Proof: Suppose  $x \in \hat{K}$ . Then  $0 \in \hat{K}(x)$  by lemma B1, and by proposition A1,  $0 \in K(x)$ . Hence by (3.2),  $x \in K$ . Conversely, suppose  $x \in K$  and each  $u_i$  is locally pseudoconcave at  $x$ . Let  $\bar{\lambda} = \min_{i \in N} \{\lambda_i\} > 0$ , and let  $U = \{x + \lambda v \mid v \in B, 0 \leq \lambda < \bar{\lambda}\}$ . If for some  $x + \lambda v \in U$ ,  $M = \{i \in N \mid u_i(x + \lambda v) > u_i(x)\} \in \mathcal{W}$ , then local pseudoconcavity and superadditivity imply  $M \subset M(x, v) \in \mathcal{W}$ . This contradiction to  $x \in K$  shows  $x$  is locally undominated in  $U$ , so that  $x \in \hat{K}$ .

Thus propositions true for elements of  $K$  are true for elements of  $\hat{K}$ , and local pseudoconcavity is sufficient for the converse.

Proposition B2:  $K$  is closed.

Proof: Let  $\{x_t\}$  be a sequence in  $K$  converging to  $\bar{x}$ . Let  $v_t = 0 \in K(x_t)$ . Now apply the first half of the proof of proposition A2.

## APPENDIX C

In this appendix  $F(x)$ , defined in (6.2), is shown to be uppersemicontinuous. To do this, suppose  $\{x_k\}$  and  $\{y_k\}$  are two sequences in  $E^m$  such that  $x_k \rightarrow \bar{x}$ ,  $y_k \in F(x_k)$ , and  $y_k \rightarrow \bar{y}$ . Then  $\bar{y} \in F(\bar{x})$  must be shown.

If  $\bar{x} \in K \cup \bar{L}$ , the proof is trivial because each  $\|y_k\| \leq S$  implies that  $\|\bar{y}\| \leq S$ , which shows by (6.2) that  $\bar{y} \in F(\bar{x})$ . So suppose  $\bar{x} \in J \setminus K$ .

Because  $J \setminus K$  is an open set,  $x_k \in J \setminus K$  for large  $k$ . Hence

$v_k = \frac{y_k}{\|y_k\|}$  is contained in  $K(x_k)$  for large  $k$ , by (6.2). Inspection

of the first paragraph of the proof to proposition A2 now reveals that it proves  $\bar{v} \in K(\bar{x})$ , where  $v_k \rightarrow \bar{v} = \frac{\bar{y}}{\|\bar{y}\|}$ . Since

$s \leq \|\bar{y}\| \leq S$  because  $s \leq \|y_k\| \leq S$  for large  $k$ , this shows that

$\bar{y} \in F(\bar{x})$ .

## FOOTNOTES

1. Median-like symmetry conditions are discussed, for example, in Davis, DeGroot and Hinich [1972], Sloss [1973], Hoyer and Mayer [1975], and Calvert [1977]. The more explicit pairwise symmetry conditions necessary in majority rule are discussed in Plott [1967], McKelvey and Wendell [1975], Matthews [1977], and Slutsky [1977].
2. See Matthews [1976] for a variety of other interpretations of a similar model applied to the special case of electoral competition.
3. Notice that this does not say that  $c'(t) \in K(c(t))$ . Hence it is not necessarily true that a status quo moving along  $c$  is shifting in undominated directions. This kind of path is discussed later.
4. Notice that (4.5iii) implies  $\|c'(t)\| = 1$ . Hence the local cycling property implies the existence of a nondegenerate path from  $x$  to  $x$  that stays near  $x$ , which accounts for the name "local cycling."
5. See Schofield [1977b] for a very different argument for considering  $L$  as a solution set.
6. When  $N$  is not finite, neither  $\hat{K} \subset K$  or  $K \subset \hat{K}$  is true in general, even assuming local pseudoconcavity. Pseudoconcavity, however, implies both  $K \subset \hat{K}$  and  $\hat{K} \subseteq \{\text{globally undominated points}\}$ . See Calvert [1977] and Sloss [1973] for further discussion of  $K$  and  $\hat{K}$  when  $N$  is arbitrarily large.

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